JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 135, 165–177 (1988)

Interpolation Formulas for Entire Functions of Exponential Type and Some Applications

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Submitted by R. P. Boas

Received December 15, 1986

1. INTRODUCTION

For entire functions f of exponential type τ (that is, f satisfying an inequality of the form $|f(z)| \leqslant A(\varepsilon) \exp(\tau+\varepsilon)|z|$ for each positive but for no negative ε) it is possible to reconstruct the function from its values at the integers if $\tau \leqslant \pi$, f is $o(|x|)$ on the real axis and bounded at the integers. In fact, we have the following interpolation formula due to Valiron ([8], see also [1, p. 221]).

THEOREM. *If f is an entire function of exponential type at most π, $|f(x)| = o(|x|)$ $(|x| \to \infty)$ and f is bounded at the integers, f has the representation*

$$f(z) = \sin \pi z \left[\frac{f'(0)}{\pi} + \frac{f(0)}{\pi z} + z \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n f(n)}{n\pi(z-n)}\right]. \tag{1.1}$$

The formula (1.1) also holds without the growth restriction $|f(x)| = o(|x|)$ if f is of exponential type less than π. This was proved by Tschakaloff [7].

In this work we extend Valiron's interpolation formula in such a way that an entire function f of exponential type at most $m\pi$ (m a positive integer) can be reconstructed from the values of $f, f', ..., f^{(m-1)}$ at the integers if f is $o(|x|)$ on the real axis and $f, f', ..., f^{(m-1)}$ are bounded at the integers. For example, we prove the following

THEOREM. *If f is an entire function of exponential type at most 2π,*

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$|f(x)| = o(|x|)$ $(|x| \to \infty)$ *and* f *and* f' *are bounded at the integers*, f *has the representation*

$$f(z) = \sin^2 \pi z \left[\frac{f''(0)}{2\pi^2} + \frac{f(0)}{3} - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{f(n)}{\pi^2 n^2} + \sum_{n=-\infty}^{\infty} \frac{f(n)}{\pi^2 (z-n)^2} \right.$$
$$\left. + \frac{f'(0)}{\pi^2 z} + z \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{f'(n)}{n\pi^2 (z-n)} \right].$$

As an application of our formulas we give a new proof of Korevaar's ([4], see also [1, p. 202]) generalization of Cartwright's theorem ([3], see also [1, p. 180]); that is, an entire function of exponential type less than $m\pi$ is bounded on the real axis if $f, f', ..., f^{(m-1)}$ are bounded at the integers. Finally we use our formulas to derive the partial fraction expansions of the meromorphic functions $\pi^m/\sin^m \pi z$.

2. Some Preliminaries

In this section we state some preliminaries that are frequently used in the text.

First of all we need the Maclaurin series of the powers of $\sin \pi z$. Writing $\sin \pi z = \frac{1}{2i}[e^{i\pi z} - e^{-i\pi z}]$, using the binomial equation, and expanding each term into its Maclaurin series we get for positive integers m

$$\sin^m \pi z = m!(\pi z)^m \sum_{n=0}^{\infty} \alpha_{mn} (\pi z)^{2n}, \tag{2.1}$$

where

$$\alpha_{mn} = \frac{(-1)^n 4^n T(2n+m, m)}{(2n+m)!}. \tag{2.2}$$

The numbers $T(k, m)$ (k, m nonnegative integers with $m \leqslant k$) are called central factorials. For definition and some properties see, for example, [2, pp. 11–12] or [6, pp. 212–217, 233–236].

Furthermore we need the following well-known formula. If N is a positive integer, then

$$\sum_{n=1}^{\infty} \frac{N}{n(n+N)} = \sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + o(1), \tag{2.3}$$

where γ is Euler's constant.

3. Construction of Functions with Assigned Values

In the following let m be always a positive integer and for $k=0, 1, ..., m-1$ let $(a_{kn})_{n=-\infty}^{\infty}$ be bounded sequences of complex numbers. Then we construct entire functions $F=F_m$ of exponential type at most $m\pi$ such that

$$F^{(k)}(n)=a_{kn} \qquad (n\in\mathbb{Z}, k=0, 1, ..., m-1). \tag{3.1}$$

We do this in two steps.

3.1. First we construct entire functions $A_{mk}=A_k$ $(k=0, 1, ..., m-1)$ of exponential type at most $m\pi$ such that

$$A_k^{(l)}(\nu)=\delta_{kl}\delta_{\nu 0} \qquad (\nu\in\mathbb{Z}, l=0, 1, ..., m-1), \tag{3.2}$$

where δ_{kl} denotes the Kronecker symbol.

For that purpose we define the following functions.

$$A_{2\mu,2\kappa}(z)=\sum_{\nu=1}^{\mu-\kappa} C_\nu^{(2\mu,2\kappa)}\frac{\sin^{2\mu}\pi z}{(\pi z)^{2\nu}} \qquad (\mu\geqslant 1, \kappa=0, 1, ..., \mu-1), \tag{3.3}$$

$$A_{2\mu,2\kappa+1}(z)=\sum_{\nu=1}^{\mu-\kappa} C_\nu^{(2\mu,2\kappa+1)}\frac{\sin^{2\mu}\pi z}{(\pi z)^{2\nu-1}} \qquad (\mu\geqslant 1, \kappa=0, 1, ..., \mu-1), \tag{3.4}$$

$$A_{2\mu-1,2\kappa}(z)=\sum_{\nu=1}^{\mu-\kappa} C_\nu^{(2\mu-1,2\kappa)}\frac{\sin^{2\mu-1}\pi z}{(\pi z)^{2\nu-1}} \qquad (\mu\geqslant 1, \kappa=0, 1, ..., \mu-1), \tag{3.5}$$

$$A_{2\mu-1,2\kappa+1}(z)=\sum_{\nu=1}^{\mu-\kappa-1} C_\nu^{(2\mu-1,2\kappa+1)}\frac{\sin^{2\mu-1}\pi z}{(\pi z)^{2\nu}} \qquad (\mu\geqslant 2, \kappa=0, 1, ..., \mu-2), \tag{3.6}$$

where $C_\nu^{(m,k)}$ are real numbers satisfying the following systems of linear equations:

$$C_{\mu-\kappa}^{(2\mu,2\kappa)}=\frac{1}{\pi^{2\kappa}(2\kappa)!}, \qquad \sum_{\nu=0}^{n-\kappa}\alpha_{2\mu,\nu}C_{\mu-n+\nu}^{(2\mu,2\kappa)}=0 \qquad (n=\kappa+1, ..., \mu-1), \tag{3.7}$$

$$C_{\mu-\kappa}^{(2\mu,2\kappa+1)}=\frac{1}{\pi^{2\kappa+1}(2\kappa+1)!}, \qquad \sum_{\nu=0}^{n-\kappa}\alpha_{2\mu,\nu}C_{\mu-n+\nu}^{(2\mu,2\kappa+1)}=0 \qquad (n=\kappa+1, ..., \mu-1), \tag{3.8}$$

$$C_{\mu-\kappa}^{(2\mu-1,2\kappa)}=\frac{1}{\pi^{2\kappa}(2\kappa)!}, \quad \sum_{\nu=0}^{n-\kappa}\alpha_{2\mu-1,\nu}C_{\mu-n+\nu}^{(2\mu-1,2\kappa)}=0$$
$$(n=\kappa+1, ..., \mu-1), \tag{3.9}$$

$$C_{\mu-\kappa-1}^{(2\mu-1,2\kappa+1)}=\frac{1}{\pi^{2\kappa+1}(2\kappa+1)!}, \quad \sum_{\nu=0}^{n-\kappa}\alpha_{2\mu-1,\nu}C_{\mu-n+\nu-1}^{(2\mu-1,2\kappa+1)}=0$$
$$(n=\kappa+1, ..., \mu-2) \tag{3.10}$$

with $\alpha_{m\nu}$ defined by (2.2).

Then we have

THEOREM 3.1. *The functions A_{mk} defined by* (3.3)–(3.6) *are entire functions of exponential type $m\pi$ satisfying* (3.2).

Proof. It is clear that the A_{mk} are entire and of exponential type $m\pi$, so that we have only to verify (3.2). We do this in the case that m and k are even because the other three cases can be handled in the same manner.

Evidently we have (3.2) for all $\nu \neq 0$ and for $\nu=0$ if l is odd. This holds for arbitrary C_ν. Now we choose C_ν such that (3.2) also holds for even l. For that purpose we expand $A_{2\mu,2\kappa}$ into its Maclaurin series (using (2.1))

$$\begin{aligned}A_{2\mu,2\kappa}(z)&=\sum_{\nu=1}^{\mu-\kappa}C_\nu\sum_{n=0}^{\infty}(2\mu)!\,\alpha_{2\mu,n}(\pi z)^{2n+2\mu-2\nu}\\&=\sum_{n=\kappa}^{\mu-1}\left(\sum_{\nu=0}^{n-\kappa}(2\mu)!\,\alpha_{2\mu,\nu}C_{\mu-n+\nu}\right)(\pi z)^{2n}+O(z^{2\mu}).\end{aligned}$$

We notice that $A^{(2l)}(0)=0$ is fulfilled for $l=0, 1, ..., \kappa-1$, and the requirement $A^{(2\kappa)}(0)=1$ and $A^{(2l)}(0)=0$ for $l=\kappa+1, ..., \mu-1$ gives Eqs. (3.7).

3.2. Now we are ready to construct the desired functions. We first put

$$B_{mkn}(z)=B_{kn}(z)=A_{mk}(z-n) \qquad (n\in\mathbb{Z}, k=0, 1, ..., m-1). \tag{3.11}$$

Then we have by (3.2)

$$B_{kn}^{(l)}(\nu)=\delta_{kl}\delta_{\nu n} \qquad (\nu\in\mathbb{Z}, l=0, 1, ..., m-1). \tag{3.12}$$

With these notations we define the functions (interpolation operators) $F_m=F_m(\cdot\,; a_{0n}, ..., a_{m-1,n})$ as follows.

$$F_{2\mu}(z) = \sum_{j=0}^{\mu-1} \left(\sum_n B_{2j,n}(z)\, a_{2j,n} + B_{2j+1,0}(z)\, a_{2j+1,0} \right.$$
$$+ \sum_{n \neq 0} \frac{1}{n} [z B_{2j+1,n}(z)\, a_{2j+1,n}$$
$$\left. - 2j B_{2j,n}(z)\, a_{2j-1,n}] \right), \tag{3.13}$$

and

$$F_{2\mu-1}(z) = \sum_{j=0}^{\mu-1} \left(B_{2j,0}(z)\, a_{2j,0} + z \sum_{n \neq 0} \frac{1}{n} B_{2j,n}(z)\, a_{2j,n} \right)$$
$$+ \sum_{j=0}^{\mu-2} \left(\sum_n B_{2j+1,n}(z)\, a_{2j+1,n} \right.$$
$$\left. - \sum_{n \neq 0} (2j+1) \frac{1}{n} B_{2j+1,n}(z)\, a_{2j,n} \right), \tag{3.14}$$

where $\sum_n$ means that n runs over all integers and $\sum_{n \neq 0}$ that n runs over all nonzero integers.

Then we have

THEOREM 3.2. *The functions F_m defined by* (3.13) *and* (3.14) *are entire functions of exponential type at most $m\pi$ satisfying* (3.1) *and*

$$|F_m(x)| = o(|x|) \qquad (|x| \to \infty). \tag{3.15}$$

Proof. Since the series in (3.13) and (3.14) converges uniformly in any compact set excluding the integers (the terms are dominated by $O(1/n^2)$), and the singularities at the integers are removable the functions F_m are entire and evidently of exponential type at most $m\pi$.

If $m = 2\mu$ is even, we have for $k = 0, 1, ..., m-1$

$$F_{2\mu}^{(k)}(z) = \sum_{j=0}^{\mu-1} \left(\sum_n B_{2j,n}^{(k)}(z)\, a_{2j,n} + B_{2j+1,0}^{(k)}(z)\, a_{2j+1,0} \right.$$
$$+ \sum_{n \neq 0} \frac{1}{n} [z B_{2j+1,n}^{(k)}(z)\, a_{2j+1,n} + k B_{2j+1,n}^{(k-1)}(z)\, a_{2j+1,n}$$
$$\left. - 2j B_{2j,n}^{(k)}(z)\, a_{2j-1,n}] \right).$$

Putting $z = \nu$ and using (3.12) gives (3.1). For $l = 2\mu - 1$ odd (3.1) is verified in the same way.

To prove (3.15) it suffices to estimate the following terms

$$|B_{j0}(x)|, \qquad \sum_{n} \left| \frac{\sin^m \pi x}{(x-n)^{2v}} \right| \qquad \text{and} \qquad \sum_{n \neq 0} \left| \frac{x \sin^m \pi x}{n(x-n)^{2v-1}} \right|.$$

The first term is $O(1/|x|)$, and the second is $O(1)$, since it is periodic with period 1. For the third term it suffices to give an estimate for $m = v = 1$ and $x > 0$. Let N be a large positive integer, Q_N the square centered at N with sides of length 1, S the half strip $\{z\colon |\operatorname{Im} z| \leqslant \frac{1}{2},\ \operatorname{Re} z \geqslant 2\}$, and put

$$g_n(z) = \frac{z \sin \pi z}{n(z-n)} \qquad \text{and} \qquad g(z) = \sum_{n \neq 0} |g_n(z)|.$$

We show

$$g(z) \leqslant K \log N \qquad (z \in Q_N), \tag{3.16}$$

where K is a constant independent of N. Then we have $g(z) = O(\log |z|)$ $(|z| \to \infty)$ in S, and thus on the positive real axis. By the maximum principle it is enough to have (3.16) on ∂Q_N. Since $|\sin \pi z| \leqslant M$ on S, we get for $z \in \partial Q_N$

$$g(z) \leqslant 2M \left[2 + \sum_{n=1}^{\infty} \frac{N}{(n+N)(n-\frac{1}{2})} + \sum_{n=1}^{N-1} \frac{N}{(N-n)(n-\frac{1}{2})} + \sum_{n=N+1}^{\infty} \frac{N}{(n-N)(n-\frac{1}{2})} \right].$$

Now (2.3) gives (3.16).

Remark. We even proved a stronger estimate than (3.15), namely

$$|F_m(x)| = O(\log |x|) \qquad (|x| \to \infty),$$

but we shall not need it. This estimate is best possible as can be seen by the example

$$F(z) = \sum_{n=1}^{\infty} \frac{z \sin \pi z}{n(z+n)}.$$

4. The Interpolation Formulas

Now we can prove the desired interpolation formulas. For that purpose we need the following uniqueness theorem.

THEOREM 4.1 (see [2, p. 47]). *If f is an entire function of exponential type at most $m\pi$ such that $f^{(k)}(n)=0$ for all integers n and $k=0, 1, ..., m-1$, and $|f(x)|=O(|x|^p)$ $(|x|\to\infty)$ for some $p\geqslant 0$, then*

$$f(z)=P(z)\sin^m \pi z,$$

where P is a polynomial of degree at most p.

In particular, if $|f(x)|=o(|x|)$ $(|x|\to\infty)$, *then*

$$f(z)=c\sin^m \pi z,$$

where c is a complex constant.

Let f be an entire function of exponential type at most $m\pi$ and $f, f', ..., f^{(m-1)}$ bounded at the integers. Then we put

$$a_{kn}=f^{(k)}(n) \qquad (n\in\mathbb{Z}, k=0, 1, ..., m-1),$$

and

$$F_m(z; f)=F_m(z; a_{0n}, ..., a_{m-1,n}), \tag{4.1}$$

where $F_m(\cdot\,; a_{0n}, ..., a_{m-1,n})$ is defined by (3.13) and (3.14). With this notation we have

THEOREM 4.2. *If f is an entire function of exponential type at most $m\pi$,*

$$|f^{(k)}(n)|\leqslant K \qquad (n\in\mathbb{Z}, k=0, 1, ..., m-1), \tag{4.2}$$

and

$$|f(x)|=o(|x|) \qquad (|x|\to\infty), \tag{4.3}$$

then we have the interpolation formula

$$f(z)=F_m(z; f)+c\sin^m \pi z, \tag{4.4}$$

where $F_m(\cdot\,; f)$ is defined by (4.1) *and*

$$c=\frac{f^{(m)}(0)-F_m^{(m)}(0; f)}{\pi^m m!}. \tag{4.5}$$

Proof. By (4.2) and Theorem 3.2 F_m is an entire function of exponential type at most $m\pi$ satisfying (3.15) and $F_m^{(k)}(n; f)=f^{(k)}(n)$ for all integers n and $k=0, 1, ..., m-1$. We put

$$g(z)=f(z)-F_m(z; f). \tag{4.6}$$

Then g is an entire function of exponential type at most $m\pi$, and $|g(x)| = o(|x|)$ $(|x| \to \infty)$ by (3.15) and (4.3). Using Theorem 4.1 we get

$$g(z) = c \sin^m \pi z, \tag{4.7}$$

where c is a complex constant, and (4.4) is proved. Taking the mth derivative of g at $z = 0$ and using (2.1) we get (4.5) by combining (4.6) and (4.7).

Remarks. (1) For $m = 1$ Theorem 4.2 implies Valiron's interpolation formula (1.1) ([8], see also [1, p. 221]).

(2) The Theorems 4.1 and 4.2 hold under the somewhat weaker condition $\max\{h_f(\pi/2), h_f(-\pi/2)\} \leqslant m\pi$, where h_f is the indicator function of f, but it is well known that this condition together with (4.3) implies $h_f(\theta) \leqslant m\pi\, |\sin \theta|$, and thus $\tau_f \leqslant m\pi$.

(3) If (4.3) is replaced by the more general condition $|f(x)| = O(|x|^p)$ $(|x| \to \infty)$ for some $p > 0$, then a similar interpolation formula (4.4) holds, where c must be substituted by a polynomial P of degree at most p.

(4) It would be desirable to evaluate $F_m^{(m)}(0; f)$ in terms of $f(n)$, $f'(n), ..., f^{(m-1)}(n)$. One can do this by using (2.1). Then we get

$$\begin{aligned} F_{2\mu}^{(2\mu)}(0; f) = \pi^{2\mu}(2\mu)! \sum_{j=0}^{\mu-1} \sum_{\nu=1}^{\mu-j} C_\nu^{(2\mu, 2j)} \Bigg[\alpha_{2\mu,\nu}(2\mu)!\, f^{(2j)}(0) \\ + \sum_{n \neq 0} \frac{1}{(\pi n)^{2\nu}} \left(f^{(2j)}(n) - \frac{2j}{n} f^{(2j-1)}(n) \right) \Bigg], \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} F_{2\mu-1}^{(2\mu-1)}(0; f) = \pi^{2\mu-1}(2\mu-1)! \sum_{j=0}^{\mu-2} \sum_{\nu=1}^{\mu-j-1} C_\nu^{(2\mu-1, 2j+1)} \\ \times \Bigg[\alpha_{2\mu-1,\nu}(2\mu-1)!\, f^{(2j+1)}(0) \\ + \sum_{n \neq 0} \frac{(-1)^n}{(\pi n)^{2\nu}} \left(f^{(2j+1)}(n) - \frac{2j+1}{n} f^{(2j)}(n) \right) \Bigg]. \end{aligned} \tag{4.9}$$

In [7] Tschakaloff proved that Valiron's interpolation formula also holds without the growth restriction on the real axis if f is of exponential type less than π. Using the same method we can prove that our formulas hold under a similar condition.

THEOREM 4.3. *If f is an entire function of exponential type less than $m\pi$ satisfying* (4.2), *then the interpolation formula* (4.4) *holds.*

Proof. We consider the entire function

$$g(z)=\frac{f(z)-F_m(z;f)}{\sin^m \pi z}-c.$$

Since $\sin^m \pi z$ is bounded below by 1 on the circles $C_n=\{z\in\mathbb{C}: |z|=n+\frac{1}{2}\}$ (n a nonnegative integer) the maximum principle gives that g is of exponential type.

We choose positive real numbers τ and γ such that $\tau_f<\tau<m\pi$ and $\tau[1+\gamma^2]^{1/2}<m\gamma\pi$. Then we prove

$$|g(z)|=O(\log|z|) \qquad (|z|\to\infty) \tag{4.10}$$

on the four rays $z=\pm x\pm i\gamma x$ (x real and positive). For large $|x|$ we have $|\sin^m \pi z|\geqslant 4^{-m}e^{m\gamma\pi|x|}$, and therefore

$$\left|\frac{f(z)}{\sin^m \pi z}\right|=O(\exp[\tau(1+\gamma^2)^{1/2}-m\gamma\pi]\,|x|)=o(1).$$

To prove that

$$\left|\frac{F_m(z;f)}{\sin^m \pi z}\right|=O(\log|z|)$$

it suffices to estimate the two sums

$$\sum_{n\neq 0}\frac{z}{n(z-n)^\nu} \qquad \text{for} \quad \nu\geqslant 1,$$

and

$$\sum_n \frac{1}{(z-n)^\nu} \qquad \text{for} \quad \nu\geqslant 2.$$

Note that there exists a constant $M>1$ depending only on γ such that for all large $|x|$ and all integers n

$$|z-n|\geqslant\frac{1}{M}[x^2+n^2]^{1/2}\geqslant 1.$$

Then it is enough to give an estimate of the first sum for $\nu=1$ and of the second sum for $\nu=2$. For the first sum we have

$$\left|\sum_{n\neq 0}\frac{z}{n(z-n)}\right| \leqslant \sum_{n=1}^{\infty}\frac{|z|}{n}\left(\frac{1}{|z-n|}+\frac{1}{|z+n|}\right)$$
$$\leqslant 2M\,|z|\sum_{n=1}^{\infty}\frac{1}{n(x^2+n^2)^{1/2}}$$
$$\leqslant \frac{2M\,|z|}{(1+x^2)^{1/2}}+2M\,|z|\int_1^{\infty}\frac{du}{u(x^2+u^2)^{1/2}}$$
$$\leqslant O(1)+\frac{2M\,|z|}{|x|}\log(|x|+(1+x^2)^{1/2})$$
$$=O(\log|x|)=O(\log|z|),$$

and it is easily seen that the second sum is $O(1)$. Therefore, (4.10) is proved.

Finally we consider the entire function

$$G(z)=\frac{g(z)-g(0)}{z}.$$

G is of exponential type and by (4.10) $|G(z)|=o(1)$ ($|z|\to\infty$) on the four rays $z=\pm x\pm i\gamma x$. Thus, by Phragmén–Lindelöf's theorem, G is bounded, and therefore $G\equiv 0$. Now $g(z)\equiv g(0)$ and by (4.5) $g(0)=0$.

5. Some Applications

5.1. In [4] (see also [1, p. 202]) Korevaar proved a generalization of Cartwright's theorem ([3], see also [1, p. 180]) that an entire function of exponential type less than $m\pi$ is bounded on the real axis if $f, f', ..., f^{(m-1)}$ are bounded at the integers. Korevaar's proof was by induction on m. As an application of our interpolation formulas we can give another proof of this theorem.

Theorem 5.1. *If f is an entire function of exponential type $\tau < m\pi$ satisfying* (4.2), *then*

$$|f(x)|\leqslant KM \qquad (x\in\mathbb{R}), \tag{5.1}$$

where M is a constant depending only on τ and m.

If in addition

$$\lim_{n\to\infty} f(n)=a, \quad \lim_{n\to\infty} f^{(k)}(n)=0 \qquad (k=1,..., m-1),$$

then $\lim_{x\to\infty} f(x)=a$.

Proof. We only give a sketch of the proof omitting the technical details because we proceed as in Pfluger's [5] proof of Cartwright's theorem.

Let N be an integer and p be a positive integer such that $p > m\pi/(m\pi - \tau)$, and put

$$g(z) = f(z+N) \sin^m \frac{\pi z}{p}. \tag{5.2}$$

Then g satisfies the hypotheses of Theorem 4.3 (note that $g^{(k)}(n)$ depends only on $f^{(k)}(n)$, m and τ), and thus

$$f(z+N) = \frac{F_m(z; g)}{\sin^m(\pi z/p)} + d\frac{\sin^m \pi z}{\sin^m(\pi z/p)}, \tag{5.3}$$

where

$$d = \frac{g^{(m)}(0) - F_m^{(m)}(0; g)}{\pi^m m!}.$$

Let Q be the square with corners $\pm\frac{1}{2}\pm\frac{1}{2}i$. Now we show that

$$|f(z+N)| \leqslant KM \qquad (z \in \partial Q), \tag{5.4}$$

where M depends only on τ and m but not on N. Then the maximum principle and $N \to \pm\infty$ give (5.1).

Since $(\sin^m \pi z)/(\sin^m(\pi z/p))$ is regular on Q we have

$$\left|\frac{\sin^m \pi z}{\sin^m(\pi z/p)}\right| \leqslant M_1 = M_1(m, \tau) \qquad (z \in Q).$$

From the formulas (4.8) and (4.9) for $F_m^{(m)}(0; g)$ we see that $|F_m^{(m)}(0; g)| \leqslant KM_2(m)$, and therefore $|d| \leqslant KM_3(m, \tau)$, so that the second term in (5.3) is dominated by $KM_4(m, \tau)$. For the estimate of the first term in (5.3) we note that the constants $C_\nu^{(m,k)}$ (see (3.7)–(3.10)) which occur in the formulas (3.13) and (3.14) for $F_m(z; g)$ depend only on m, and as in [5] it can be shown that the sums

$$\sum_n \frac{1}{|z-n|^{2\nu}} \qquad \text{and} \qquad \sum_{n \neq 0} \frac{|z|}{|n(z-n)^{2\nu-1}|}$$

are dominated on ∂Q by constants depending only on m (since $2\nu \leqslant m$). That gives (5.4).

For the proof of the addition we choose with $\varepsilon > 0$ a positive integer $N(\varepsilon)$ such that (we can assume $a = 0$)

$$|f^{(k)}(n)| < \varepsilon \qquad (n > N(\varepsilon),\ k = 0, 1, ..., m-1),$$

and

$$\sum_{n=N(\varepsilon)}^{\infty} \frac{1}{n^2} < \varepsilon.$$

Furthermore let p be as befor, N an integer with $N > 2N(\varepsilon)$, and g as in (5.2), so that we have again (5.3). Then it remains to show

$$|f(z+N)| < C\varepsilon \qquad (z \in \partial Q), \tag{5.5}$$

where C is a constant depending only on K, m, and τ but not on N, which gives

$$|f(x)| < C\varepsilon \qquad (x > 2N(\varepsilon)).$$

Proceeding as in [5] (5.5) can be proved similar to (5.4) with the difference that the sums over n occurring in the formulas for $F_m(z; g)$ and $F_m^{(m)}(0; g)$ must be split at the term $N(\varepsilon)$.

5.2. As another application of our formulas we can give the partial fraction expansions of the meromorphic functions $\pi^m/\sin^m \pi z$. This is done by applying (4.4) to the constant function $f \equiv 1$ and deviding by $(\sin^m \pi z)/\pi^m$. Then we get

$$\frac{\pi^{2\mu}}{\sin^{2\mu} \pi z} = \sum_n \left[\sum_{\nu=1}^{\mu} C_\nu^{(0,2\mu)} \frac{\pi^{2(\mu-\nu)}}{(z-n)^{2\nu}} \right],$$

and

$$\frac{\pi^{2\mu-1}}{\sin^{2\mu-1} \pi z} = \sum_{\nu=1}^{\mu} C_\nu^{(0,2\mu-1)} \frac{\pi^{2(\mu-\nu)}}{z^{2\nu-1}} + \sum_{n \neq 0} (-1)^n \left[\sum_{\nu=1}^{\mu} C_\nu^{(0,2\mu-1)} \frac{\pi^{2(\mu-\nu)}}{(z-n)^{2\nu-1}} + C_1^{(0,2\mu-1)} \frac{\pi^{2\mu-2}}{n} \right],$$

where the constants $C_\nu^{(0,m)}$ are defined by (3.7) and (3.8).

REFERENCES

1. R. P. BOAS, JR., "Entire Functions," Academic Press, New York, 1954.
2. R. BRÜCK, Identitätssätze für ganze Funktionen vom Exponentialtyp, *Mitt. Math. Sem. Giessen*, Heft 168 (1984).
3. M. L. CARTWRIGHT, On certain integral functions of order 1, *Quart. J. Math. Oxford Ser. (1)* **7** (1936), 46–55.
4. J. KOREVAAR, Functions of exponential type bounded on sequences of points, *Ann. Soc. Polon. Math.* **22** (1949), 207–234.
5. A. PFLUGER, On analytic functions bounded on the lattice points, *Proc. London Math. Soc. (2)* **42** (1937), 305–315.
6. J. RIORDAN, "Combinatorial Identities," Wiley, New York, 1968.
7. L. TSCHAKALOFF, Zweite Lösung der Aufgabe 105, *Jahresber. Deutsch. Math.-Verein.* **43** (1934), 11–13.
8. G. VALIRON, Sur la formule d'interpolation de Lagrange, *Bull. Sci. Math. (2)* **49** (1925), 181–192, 203–224.